



Total domination and the Caccetta–Häggkvist conjecture

Patrick St-Louis^a, Bernard Gendron^{b,c}, Alain Hertz^{d,e,*}

^a GIRO inc., 75, rue de Port-Royal Est, Montréal, Québec, H3L 3T1 Canada

^b Département d'informatique et de recherche opérationnelle, Canada

^c CIRRELT, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Québec, H3C 3J7 Canada

^d Département de mathématiques et de génie industriel, École Polytechnique de Montréal, Canada

^e GERAD, C.P. 6079, succ. Centre-ville, Montréal, Québec, H3C 3A7 Canada

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ABSTRACT

A total dominating set in a digraph G is a subset W of its vertices such that every vertex of G has an immediate successor in W . The total domination number of G is the size of the smallest total dominating set. We consider several lower bounds on the total domination number and conjecture that these bounds are strictly larger than $g(G) - 1$, where $g(G)$ is the number of vertices of the smallest directed cycle contained in G . We prove that these new conjectures are equivalent to the Caccetta–Häggkvist conjecture which asserts that $g(G) - 1 < \frac{n}{r}$ in every digraph on n vertices with minimum outdegree at least $r > 0$.

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1. Introduction

Let $G = (V, A)$ be a digraph with vertex set V and arc set A ; we assume that G has no multiple arcs, no loops and no directed cycles of length 2. The girth of G , denoted $g(G)$, is the number of vertices of the smallest directed cycle in G . Let $\delta^+(G)$ denote the minimum outdegree of G . In 1978, Caccetta and Häggkvist [1] proposed the following conjecture.

Conjecture 1. *If G is a digraph with n vertices and $\delta^+(G) \geq r > 0$, then $g(G) \leq \lceil \frac{n}{r} \rceil$.*

This conjecture has been verified for values of r up to 5 [1–3] and for $n \geq 2r^2 - 3r + 1$ [4]. One approach to verify the validity of this conjecture is to show that if G is a digraph with n vertices and $\delta^+(G) \geq r$, then there is a directed cycle in G of length at most $\frac{n}{r} + c$ for some small c . This has been proved for $c = 2500$ [5], $c = 304$ [6] and $c = 73$ [7]. In 2006, a workshop was held in Palo Alto, California, with the Caccetta–Häggkvist conjecture as its central subject. A summary of the results (and much more) was published by Sullivan [8]. More recent developments can be found in [9,10].

Let $N_G^+(v)$ denote the set of immediate successors of a vertex $v \in V$. A *total dominating set* in a digraph G is a subset W of its vertices such that $N_G^+(v) \cap W \neq \emptyset$ for every vertex $v \in V$. The total domination number of G , denoted $\text{td}(G)$ is the size of the smallest total dominating set of G . Total domination was introduced by Cockayne et al. [11] and is now a well studied concept in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [12,13]. Recent developments can also be found in [14–19].

We assume $\delta^+(G) > 0$, else G does not contain any total dominating set. Finding a total dominating set of size $\text{td}(G)$ can be modeled as the assignment of a weight $\omega_v \in \{0, 1\}$ to every vertex $v \in V$ so that $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every $v \in V$ and $\sum_{v \in V} \omega_v$ is minimized. Note that $\sum_{u \in V} \omega_u \geq \sum_{u \in N_G^+(v)} \omega_u \geq 1$, for any $v \in V$, i.e., $\text{td}(G)$ is at least 1. Better lower bounds on $\text{td}(G)$ can be obtained by considering real values for the weights:

* Corresponding author at: Département de mathématiques et de génie industriel, École Polytechnique de Montréal, Canada.

E-mail addresses: Patrick.StLouis@giro.ca (P. St-Louis), Bernard.Gendron@cirrelt.ca (B. Gendron), Alain.Hertz@gerad.ca (A. Hertz).

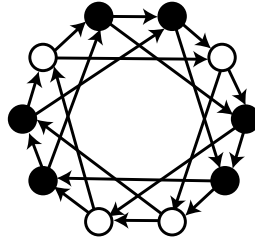


Fig. 1. A digraph with $td(G) = \frac{n}{r} + 1$.

- We denote by $tdf(G)$ the minimum total weight $\sum_{v \in V} \omega_v$ so that $\omega_v \in [0, 1]$ and $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every $v \in V$.
- By imposing $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ only for vertices with a strictly positive weight and by requiring that $\sum_{v \in V} \omega_v \geq 1$, one gets a lower bound on $tdf(G)$. More precisely, we denote by $tdfr(G)$ the minimum total weight $\sum_{v \in V} \omega_v$ so that $\omega_v \in [0, 1]$ for every $v \in V$, $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every v with $\omega_v > 0$, and $\sum_{v \in V} \omega_v \geq 1$.

It follows from the above definitions that $tdfr(G) \leq tdf(G) \leq td(G)$. As shown in Section 2, we also have $g(G) \leq td(G)$. We state the two following conjectures.

Conjecture 2. *The relation $g(G) - 1 < tdf(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

Conjecture 3. *The relation $g(G) - 1 < tdfr(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

We prove in this paper that the two new conjectures are equivalent to [Conjecture 1](#) of Caccetta and Häggkvist. In Section 2, we present mathematical programming formulations that can be used to compute $td(G)$ and its lower bounds $g(G)$, $tdf(G)$ and $tdfr(G)$. We use these formulations to prove the equivalence of the three conjectures. In Section 3, we show how to reformulate [Conjecture 3](#) using Lagrangian relaxation techniques.

2. Mathematical programming formulations

The adjacency matrix A of a digraph G is the $n \times n$ matrix where $a_{ij} = 1$ if there is an arc from i to j , and $a_{ij} = 0$ otherwise. We denote by e the vector with n entries equal to 1. The problem of determining $td(G)$ will be denoted $P_{td}(G)$ and can be modeled as an integer programming model as follows:

$$\begin{aligned} td(G) &= \text{Min } e^T \omega \\ \text{s.t. } A\omega &\geq e, \\ \omega &\in \{0, 1\}^n. \end{aligned} \tag{1}$$

Determining $g(G)$ can be viewed as the selection of the smallest subset W of vertices such that $N_G^+(v) \cap W \neq \emptyset$ for every vertex v in W . This problem, denoted $P_g(G)$, can be modeled with the following integer programming model, where constraints (3) ensure that at least one vertex is selected in W :

$$\begin{aligned} g(G) &= \text{Min } e^T \omega \\ \text{s.t. } A\omega &\geq \omega, \\ e^T \omega &\geq 1, \\ \omega &\in \{0, 1\}^n. \end{aligned} \tag{2}$$

$$e^T \omega \geq 1, \tag{3}$$

Property 4. *The relation $g(G) \leq td(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

Proof. Since the inequalities $A\omega \geq e$ imply $e^T \omega \geq 1$, one can add constraints (3) to the computation of $td(G)$ without modifying the optimal value of $P_{td}(G)$. Since $\omega \leq e$, constraints (1) are stronger than (2), which proves that $g(G) \leq td(G)$. \square

To prove the validity of [Conjecture 1](#) it would have been sufficient to show that $td(G) < \frac{n}{r} + 1$, since this would imply $g(G) - 1 \leq td(G) - 1 < \frac{n}{r}$, which is equivalent to $g(G) \leq \lceil \frac{n}{r} \rceil$. There are however digraphs for which $td(G) \geq \frac{n}{r} + 1$. For example, it is not difficult to verify that the digraph in [Fig. 1](#) satisfies $n = 10$, $r = \delta^+(G) = 2$ and $6 = td(G) = \frac{n}{r} + 1$, the black vertices corresponding to a total dominating set of minimum size.

The problem of computing $tdf(G)$, denoted $P_{tdf}(G)$, can be modeled by relaxing the integrality constraints in $P_{td}(G)$:

$$\begin{aligned} tdf(G) &= \text{Min } e^T \omega \\ \text{s.t. } A\omega &\geq e, \\ \omega &\geq 0. \end{aligned} \tag{1}$$

By imposing $A\omega \geq \lceil \omega \rceil$, we require $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every vertex v with $\omega_v > 0$. Hence, the problem $P_{\text{tdfr}}(G)$ of computing $\text{tdfr}(G)$ can be modeled as follows:

$$\begin{aligned} \text{tdfr}(G) &= \text{Min } e^T \omega \\ \text{s.t. } A\omega &\geq \lceil \omega \rceil, \end{aligned} \tag{4}$$

$$\begin{aligned} e^T \omega &\geq 1, \\ \omega &\geq 0. \end{aligned} \tag{3}$$

Note that if **Conjecture 3** is verified then $g(G) - 1 < \text{tdfr}(G) \leq g(G)$ for all digraphs G with $\delta^+(G) > 0$, since by setting $\omega_v = 1$ for all vertices v in a smallest directed cycle in G and $\omega_v = 0$ for the other vertices, one gets a feasible solution to $P_{\text{tdfr}}(G)$ of value $g(G)$.

Theorem 5. *Conjectures 2 and 3 are equivalent.*

Proof. If **Conjecture 3** is verified then $g(G) - 1 < \text{tdfr}(G) \leq \text{tdf}(G)$, which implies that **Conjecture 2** is verified also.

So assume that **Conjecture 3** is not verified and let G be a smallest counter-example (in terms of number of vertices). It remains to prove that **Conjecture 2** is also not verified. Let ω^* denote an optimal solution to $P_{\text{tdfr}}(G)$, and let G' denote the sub-digraph of G induced by all vertices v with weight $\omega_v^* > 0$. Constraints (4) impose that each vertex in G' has at least one successor in G' . Hence G' contains at least one directed cycle and we obviously have $g(G') \geq g(G)$. Also, $\text{tdfr}(G') \leq \text{tdfr}(G)$ since the restriction of ω^* to G' is a feasible solution to $P_{\text{tdfr}}(G')$. In summary, $g(G') - 1 \geq g(G) - 1 \geq \text{tdfr}(G) \geq \text{tdfr}(G')$. Since G is the smallest counter-example to **Conjecture 3**, we necessarily have $G' = G$, which means that $\omega_v^* > 0$ for all vertices in G . Hence, ω^* is a feasible solution to $P_{\text{tdf}}(G)$, which means that $\text{tdf}(G) = \text{tdfr}(G)$ and G is therefore also a counter-example to **Conjecture 2**. \square

Theorem 6. *Conjectures 1 and 2 are equivalent.*

Proof. Let G be a digraph with n vertices and consider any real number r such that $0 < r \leq \delta^+(G)$. The vector ω defined by $\omega_v = \frac{1}{r}$ for all $v \in V$ is a feasible solution to $P_{\text{tdf}}(G)$, which means that $\text{tdf}(G) \leq e^T \omega = \frac{n}{r}$. Hence, if **Conjecture 2** is verified, then $g(G) - 1 < \text{tdf}(G) \leq \frac{n}{r}$ for all digraphs G with n vertices and $\delta^+(G) \geq r > 0$, which implies that **Conjecture 1** is verified also.

So assume that **Conjecture 2** is not verified. It remains to show that **Conjecture 1** is also not verified. Let G be a smallest counter-example to **Conjecture 2** (in terms of number of vertices), and let ω^* be any optimal basic solution to $P_{\text{tdf}}(G)$. We necessarily have $\omega_v^* > 0$ for all $v \in V$, otherwise by using the same arguments as in the proof of the previous theorem, we can show that the sub-digraph G' induced by the vertices with $\omega_v^* > 0$ verifies $g(G') - 1 \geq g(G) - 1 \geq \text{tdf}(G) \geq \text{tdf}(G')$, which contradicts the minimality of G .

Constraints (1) can be rewritten as $A\omega - s = e$ by using slack variables $s \geq 0$. Consider the values $s^* = A\omega^* - e$ of the slack variables associated with ω^* . Since $P_{\text{tdf}}(G)$ contains n constraints, any basic solution to $P_{\text{tdf}}(G)$ has at most n variables with a strictly positive value. As shown above, $\omega_v^* > 0$ for all $v \in V$, and we therefore have $s^* = 0$, which implies $A\omega^* = e$. In other words, if we denote by $P_{\text{tdf}}^=(G)$ the linear program obtained from $P_{\text{tdf}}(G)$ by replacing inequalities (1) by equalities, we have proved that $P_{\text{tdf}}^=(G)$ and $P_{\text{tdf}}(G)$ have the same set of optimal solutions.

We now show that the determinant $\det(A)$ of matrix A is not equal to 0. If $\det(A) = 0$, then at least one of the n constraints in $P_{\text{tdf}}^=(G)$ is redundant. By removing such a constraint, the optimal value remains unchanged, while there are now $n - 1$ constraints for n variables. This means that $P_{\text{tdf}}^=(G)$ has an optimal solution ω^* (which is also optimal for $P_{\text{tdf}}(G)$) with at least one variable $\omega_v^* = 0$, a contradiction. We therefore have $\det(A) \neq 0$.

Let A_v denote the matrix obtained from A by replacing the v th column by vector e . Cramer's rule [20] states that $\omega_v^* = \frac{\det(A_v)}{\det(A)}$. Since $\omega_v^* > 0$, we can write

$$\omega_v^* = \frac{|\det(A_v)|}{|\det(A)|}.$$

We now construct a new graph \tilde{G} from G by replacing every vertex v by a set S_v of $|\det(A_v)|$ non-adjacent vertices. We put an arc from a vertex in S_u to a vertex in S_v if and only if there is an arc from u to v in G . Let $\tilde{V} = \bigcup_{v \in V} S_v$ denote the vertex set of \tilde{G} and define $\tilde{\omega}_{\tilde{v}} = \frac{1}{|\det(A)|}$ for all $\tilde{v} \in \tilde{V}$. In other words, \tilde{G} is obtained from G by replacing every vertex v of weight ω_v^* by $|\det(A_v)|$ non-adjacent copies of v of weight $\frac{1}{|\det(A)|}$. This means that the following equalities hold for every vertex $\tilde{v} \in S_v$:

$$\sum_{\tilde{u} \in N_{\tilde{G}}^+(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{u \in N_G^+(v)} \sum_{\tilde{u} \in S_u} \frac{1}{|\det(A)|} = \sum_{u \in N_G^+(v)} \frac{|\det(A_u)|}{|\det(A)|} = \sum_{u \in N_G^+(v)} \omega_u^* = 1.$$

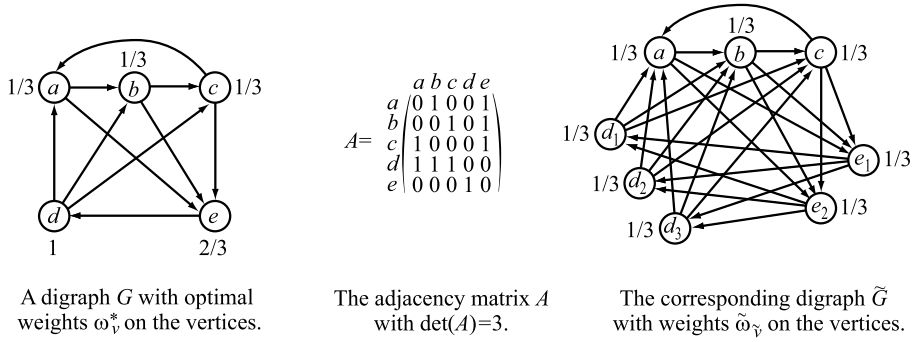


Fig. 2. A digraph G and the corresponding digraph \tilde{G} .

In addition, we have:

$$\sum_{\tilde{v} \in \tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{v \in V} \sum_{\tilde{v} \in S_v} \tilde{\omega}_{\tilde{v}} = \sum_{v \in V} \frac{|\det(A_v)|}{|\det(A)|} = \sum_{v \in V} \omega_v^* = \text{tdf}(G).$$

Hence, $\tilde{\omega}$ is a feasible solution to $P_{\text{tdf}}(\tilde{G})$ of value $\text{tdf}(G)$, which means that $\text{tdf}(\tilde{G}) \leq \text{tdf}(G)$.

Since, for every $\tilde{v} \in S_v$, $\sum_{\tilde{u} \in N_G^+(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{\tilde{u} \in N_G^+(\tilde{v})} \frac{1}{|\det(A)|} = 1$, we deduce that $|N_G^+(\tilde{v})| = |\det(A)|$, which means that $\delta^+(\tilde{G}) = |\det(A)|$. Also, since $\sum_{\tilde{v} \in \tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{v \in V} \frac{1}{|\det(A)|} = \text{tdf}(G)$, we have $|\tilde{V}| = |\det(A)| \cdot \text{tdf}(G)$. Moreover, we clearly have $g(\tilde{G}) = g(G)$.

To conclude, let $n' = |\tilde{V}|$ denote the number of vertices in \tilde{G} and let $r' = \delta^+(\tilde{G}) = |\det(A)|$. We have

$$g(\tilde{G}) - 1 = g(G) - 1 \geq \text{tdf}(G) = \frac{|\det(A)| \cdot \text{tdf}(G)}{|\det(A)|} = \frac{n'}{r'}$$

which means that \tilde{G} is a counter-example to Conjecture 1. \square

The construction described in the proof of Theorem 6 is illustrated in Fig. 2 for a digraph G with $\text{tdf}(G) = \frac{8}{3}$ and $g(G) = 3$.

3. Reformulation of Conjecture 3

Problem $P_{\text{tdfr}}(G)$ can be written as an integer linear programming model by replacing $A\omega \geq \lceil \omega \rceil$ with $A\omega \geq y \geq \omega$, $y \in \{0, 1\}^n$. Hence, an equivalent model for $P_{\text{tdfr}}(G)$ reads as follows:

$$\begin{aligned} \text{tdfr}(G) &= \text{Min } e^T \omega \\ \text{s.t. } A\omega &\geq y, & (5) \\ y &\geq \omega, & (6) \\ e^T \omega &\geq 1, & (3) \\ \omega &\geq 0, \quad y \in \{0, 1\}^n. \end{aligned}$$

If we now replace constraints (6) by $y = \omega$, we obtain an equivalent model for $P_g(G)$ since

- constraints (5) are then equivalent to constraints (2);
- constraints $\omega \geq 0$, $y \in \{0, 1\}^n$ are then equivalent to $\omega \in \{0, 1\}^n$.

In other words, a model for $P_g(G)$ can be obtained from the above model for $P_{\text{tdfr}}(G)$ by adding the constraints $y \leq \omega$. Consider now the Lagrangian relaxation of this model for $P_g(G)$ obtained by relaxing constraints $y \leq \omega$ and by penalizing their violation in the objective function. More formally, given a penalty vector $\lambda \geq 0$ with n entries, we consider the problem $P_{g_\lambda}(G)$ of computing $g_\lambda(G)$ defined as follows:

$$\begin{aligned} g_\lambda(G) &= \text{Min } e^T \omega + \lambda^T (y - \omega) \\ \text{s.t. } A\omega &\geq y, & (5) \\ y &\geq \omega, & (6) \\ e^T \omega &\geq 1 & (3) \\ \omega &\geq 0, \quad y \in \{0, 1\}^n. \end{aligned}$$

Property 7. The relations $\text{tdfr}(G) \leq g_\lambda(G) \leq g(G)$, for any $\lambda \geq 0$, $\text{tdfr}(G) = g_0(G)$ and $g(G) = g_\lambda(G)$, for any $\lambda \geq e$, hold for all digraphs G with $\delta^+(G) > 0$.

Proof. For any $\lambda \geq 0$, $P_{g_\lambda}(G)$ and $P_{\text{tdfr}}(G)$ have the same set of feasible solutions. Since $\lambda^T(y - \omega) \geq 0$ because of constraints (6), we have $\text{tdfr}(G) \leq g_\lambda(G)$. Let ω^* be an optimal solution to $P_g(G)$ and define $y^* = \omega^*$. Since (ω^*, y^*) is feasible for $P_{g_\lambda}(G)$ and $\lambda^T(y^* - \omega^*) = 0$, we have $g_\lambda(G) \leq g(G)$. For $\lambda = 0$, $P_{g_\lambda}(G)$ corresponds to $P_{\text{tdfr}}(G)$, which means that $\text{tdfr}(G) = g_0(G)$. For any $\lambda \geq e$, let (ω^*, y^*) be an optimal solution to $P_{g_\lambda}(G)$. If $\omega^* < y^*$, we can replace ω^* by y^* and remain feasible, but also optimal, since the objective would then vary by the quantity $(e - \lambda)^T(y^* - \omega^*) \leq 0$. Therefore, for any $\lambda \geq e$, there exists an optimal solution to $P_{g_\lambda}(G)$ that satisfies $\omega^* = y^*$, which means that $g(G) = g_\lambda(G)$. \square

From this result, it follows directly that $\text{tdfr}(G) = \min_{\lambda \geq 0} g_\lambda(G)$ and $g(G) = \max_{\lambda \geq 0} g_\lambda(G)$. Conjecture 3 can therefore be rewritten in the following way:

Reformulation of Conjecture 3

The relation $\max_{\lambda \geq 0} g_\lambda(G) - \min_{\lambda \geq 0} g_\lambda(G) < 1$ holds for all digraphs G with $\delta^+(G) > 0$.

References

- [1] L. Caccetta, R. Häggkvist, On minimal digraphs with given girth, *Congressus Numerantium* 21 (1978) 181–187.
- [2] Y.O. Hamidoune, A note on minimal directed graphs with given girth, *Journal of Combinatorial Theory, Series B* 43 (1987) 343–348.
- [3] C. Hoáng, B. Reed, A note on short cycles in digraphs, *Discrete Mathematics* 66 (1987) 103–107.
- [4] J. Shen, On the girth of digraphs, *Discrete Mathematics* 211 (2000) 167–181.
- [5] V. Chvátal, E. Szemerédi, Short cycles in directed graphs, *Journal of Combinatorial Theory, Series B* 35 (1983) 323–327.
- [6] T. Nishimura, Short cycles in digraphs, *Discrete Mathematics* 72 (1988) 295–298.
- [7] J. Shen, On the Caccetta–Häggkvist conjecture, *Graphs and Combinatorics* 18 (2002) 645–654.
- [8] B.D. Sullivan, A summary of results and problems related to the Caccetta–Häggkvist conjecture, 2008. [arXiv:math/0605646v1](https://arxiv.org/abs/math/0605646v1).
- [9] P. Monti, P. Penna, R. Silvestri, An equivalent version of the Caccetta–Häggkvist conjecture in an online load balancing problem, in: *WG'07 Proceedings of the 33rd International Conference on Graph-Theoretic Concepts in Computer Science*, Springer-Verlag, Berlin, Heidelberg, 2007, pp. 154–165.
- [10] A. Razborov, On the Caccetta–Häggkvist conjecture with forbidden subgraphs, 2011. [arXiv:1107.2247](https://arxiv.org/abs/1107.2247).
- [11] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs, *Networks* 10 (1980) 211–219.
- [12] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [13] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
- [14] O. Favaron, M.A. Henning, C.M. Mynhardt, J. Puech, Total domination in graphs with minimum degree three, *Journal of Graph Theory* 34 (2000) 9–19.
- [15] M.A. Henning, Graphs with large total domination number, *Journal of Graph Theory* 35 (2000) 21–45.
- [16] D.C. Fisher, Fractional dominations and fractional total dominations of graph complements, *Discrete Applied Mathematics* 122 (2002) 283–291.
- [17] J. Huang, J. Xu, The total domination and total bondage numbers of extended de Bruijn and Kautz digraphs, *Computers & Mathematics with Applications* 53 (2007) 1206–1213.
- [18] M.A. Henning, A. Yeo, A new upper bound on the total domination number of a graph, *The Electronic Journal of Combinatorics* 14 (2007) R65.
- [19] P.C.B. Lam, B. Wei, On the total domination number of graphs, *Utilitas Mathematica* 72 (2007) 223–240.
- [20] G. Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750, pp. 657–659.